Discrete Mathematics 127 (1994) 23–29
North-Holland

A degree characterisation of pancyclicity

R.E.L. Aldred and D.A. Holton

Department of Mathematics and Statistics, University of Otago, P.O. Box 56, Dunedin, New Zealand

Zhang Ke Min

Department of Mathematics, Nanjing University, Nanjing, China

Received 11 October 1990
Revised 27 September 1991

Abstract

A graph G of order n is said to be in the class $O(n-1)$ if $\deg(u)+\deg(v)\geqslant n-1$ for every pair of nonadjacent vertices $u, v\in V(G)$. We characterise the graphs in $O(n-1)$ which are pancyclic.

1. Introduction

In this paper we consider only simple graphs. Unless otherwise stated, G will have order n and vertex set $V(G)=\{1,2,3,\ldots,n\}$. We say $G\in O(p)$ if $\deg(u)+\deg(v)\geqslant p$ for every pair of nonadjacent vertices $u, v\in V(G)$. A graph is said to be pancyclic if it contains a cycle of length m for each m, $3\leqslant m\leqslant n$.

The condition $O(n)$ was introduced by Ore [3] as a sufficient condition for G to be hamiltonian. Since that time Ore's result has been strengthened in two directions. The first, due to Bondy [2], shows that the $O(n)$ condition gives us rather more than hamiltonian. In fact, it gives the following theorem.

Theorem 1.1 *If $G\in O(n)$, then G is pancyclic unless $n=2k$ and $G\cong K_{k,k}$.*

The second uses the most basic property of hamiltonian graphs, that they are 2-connected, as an extra assumption to show that Ore's $O(n)$ condition can be slightly relaxed. This result can be found in [1]. In the following theorem $K^c_{(n+1)/2}\vee K_{(n-1)/2}$ is used to denote the graph obtained by taking the join of $K^c_{(n+1)/2}$ and $K_{(n-1)/2}$ (i.e. that graph with vertex set $V=V(K^c_{(n+1)/2})\cup V(K_{(n-1)/2})$ and edge set $E=E(K^c_{(n+1)/2})$ $\cup E(K_{(n-1)/2})\cup\{(x,y)\colon x\in V(K^c_{(n+1)/2})$ and $y\in V(K_{(n-1)/2})\}$).

Correspondence to: R.E.L. Aldred, Department of Mathematics and Statistics, University of Otago, P.O. Box 56, Dunedin, New Zealand.

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SSDI 0012-365X(92)00464-F

Theorem 1.2. *Let G be a 2-connected graph. If $G \in O(n-1)$, then G is hamiltonian unless G is isomorphic to a subgraph of $K^c_{(n+1)/2} \vee K_{(n-1)/2}$.*

(Note that 2-connected $O(n-1)$ graphs are easily characterised.)

In this paper we shall characterise those $O(n-1)$ graphs which are pancyclic. To do this we will use the above results together with the following structure theorem for hamiltonian graphs. Both Theorem 1.3 and Lemma 1.4 appear in [4].

Theorem 1.3. *Let G be a hamiltonian graph with hamiltonian cycle $C=(1,2,\ldots,n,1)$. Suppose that $\deg(1)+\deg(n)\geqslant n$ with say $\deg(1)\leqslant\deg(n)$. Then G is either*

(i) *pancyclic,*

(ii) *bipartite, or*

(iii) *missing only an $(n-1)$-cycle.*

Moreover, if (iii) holds, then $\deg(n-2), \deg(n-1), \deg(2), \deg(3) < n/2$, and G has one of two possible adjacency structures near 1 and n. In the first structure, the vertices $n-2, n-1, n, 1, 2, 3$ are independent except for the edges of C, and $(n, n-3), (n, n-4), (1,4), (1,5)$. The second structure (which can occur only if $\deg(1)<\deg(n)$) is identical to the first except that $(n,3)$ is an edge in G and $(1,5)$ is not.

Lemma 1.4. *Let G be a hamiltonian graph with hamiltonian cycle $C=(1,2,\ldots,n,1)$. Suppose that $\deg(1)+\deg(n)>n$. Then G is pancyclic.*

2. Main results

We begin with an investigation of those graphs $G \in O(n-1)$ which are regular of degree $(n-1)/2$. Note that such graphs can exist only when $n \equiv 1 \pmod 4$.

Lemma 2.1. *With the exception of C_5, every $(n-1)/2$-regular graph contains a 3-cycle.*

Proof. Let G be an $(n-1)/2$-regular graph. G must be 2-connected and hence, by Theorem 1.2, G is hamiltonian. Since n is odd and G is hamiltonian, G is not bipartite. Let C_m be the shortest odd cycle in G. The minimality of C_m dictates that

every vertex

$$v \in V(C_m) \text{ has } |N(v) \cap V(G) \setminus V(C_m)| = ((n-1)/2) - 2, \tag{1}$$

every vertex

$$u \in V(G) \setminus V(C_m) \text{ has } |N(u) \cap V(C_m)| \leqslant 2. \tag{2}$$

Thus, we have

$$m(n-5)/2 \leqslant 2(n-m),$$

i.e.

$$n(m-4) \leqslant m. \tag{3}$$

But (3) can only be true when $m=3$ or $m=n=5$. □

Theorem 2.2. *With the exception of C_5, every $(n-1)/2$-regular graph is pancyclic.*

Proof. G is hamiltonian with hamiltonian cycle $C=(1,2,\ldots,n,1)$. Let C be chosen so that $(n,r+1)$ is a shortest possible chord to C(i.e. the end vertices of every chord to C are separated by at least r vertices on C). If G is not pancyclic, then there is some m, $3\leqslant m\leqslant n$, such that G contains no m-cycle. In the light of Lemma 2.1, we may assume that $4\leqslant m\leqslant n-1$ unless $G\cong C_5$. We distinguish two cases.

Case 1: $m\leqslant r+1$. Consider the adjacencies of vertices n and $m-2$. If $(N(n)\cap N(m-2))\backslash\{1\}$ contains a vertex, j say, then $(n,j,m-2,m-3,\ldots,2,1,n)$ is an m-cycle in G. Thus $(N(n)\cap N(m-2))\backslash\{1\}=\emptyset$. This observation, together with the minimality of r gives the following:

$$|(N(n)\cup N(m-2))\cap\{r+1,r+2,\ldots,n-r+m-3\}|=2((n-1)/2-2)=n-5.$$

On the other hand,

$$\begin{aligned}&|(N(n)\cup N(m-2))\cap\{r+1,r+2,\ldots,n-r+m-3\}|\\&\quad\leqslant|\{r+1,r+2,\ldots,n-r+m-3\}|=n-(2r-m+3).\end{aligned}$$

So we have $r\leqslant(m+2)/2\leqslant(r+3)/2$.

Thus $r\leqslant 3$ and the only possibility is that $r=3$ and $m=4$. In this case we see that $N(n)\cap N(2)=\{1\}$ and consequently $N(n)\cup N(2)=\{1,3,4,5,\ldots,n-2,n-1\}$. Now 2 cannot be adjacent to 5, so $5\in N(n)$. Also $6\notin N(n)$ and $7\notin N(n)$ (otherwise we get 4-cycles). Continuing in this way we find that

$$N(n)=\{1,n-1\}\cup\{4j,4j+1\colon 1\leqslant j\leqslant(n-5)/4\}$$

and

$$N(2)=\{1,3\}\cup\{4j+2,4j+3\colon 1\leqslant j\leqslant(n-5)/4\}.$$

With the above adjacencies determined, we can move on using similar arguments to conclude that $n-1\in N(4)$ and $(n,5,4,n-1,n)$ is a 4-cycle in G. This completes the proof of case 1.

Case 2: $m\geqslant r+3$ (note that $m\neq r+2$ since $(n,r+1,r,r-1,\ldots,2,1,n)$ is an m-cycle). Consider the adjacencies of the vertices n and $m-1$. Let n be adjacent to α vertices in $L=\{n,1,2,\ldots,m-2\}$. Then n is also adjacent to $((n-1)/2-1-\alpha)=(n-3)/2-\alpha$ vertices in $R=\{m,m+1,\ldots,n-r-1\}$ and also to $n-1$.

Now each vertex $i\in L$ adjacent to n excludes the vertex $i-1$ (with $n\equiv 0$), also in L, from being adjacent to $m-1$. Thus we have

$$|N(m-1)\cap L|\leqslant|L|-\alpha=m-1-\alpha. \tag{4}$$

Each neighbour j of n in R, with the possible exception of m, excludes the vertex $j+r-1$ in $R'=\{m+r, m+r+1, \ldots, n-1\}$ from being a neighbour of $m-1$ (otherwise we have the m-cycle $(n, j, j+1, \ldots, j+r-1, m-1, m-2, \ldots, r+1, n)$. Note that, in fact, if $m-1$ is adjacent to $j-r+1$, we obtain a similar m-cycle. To avoid counting the same exclusion twice, we shall count only $j+r-1$ here. Later this observation[1] will be used to strengthen the count in special circumstances. In addition to these exclusions, the minimality of r requires that $m-1$ is not adjacent to any vertex in $(R\backslash R')\backslash\{m\}$. This gives us

$$|N(m-1)\cap(R\cup R')|\leqslant|R\cup R'|-((n-3)/2-\alpha-1)-(r-1)$$

$$=(n+7)/2-m-r+\alpha. \tag{5}$$

Combining (4) and (5) we have

$$\deg(m-1)\leqslant(m-\alpha-1)+((n+7)/2-m-r+\alpha)=(n+5)/2-r.$$

But $\deg(m-1)=(n-1)/2$, so we require that $(n+5)/2-r\geqslant(n-1)/2$, i.e. $r\leqslant 3$.

Case 2.1: When $r=3$, equality must hold in (4) and (5), so $m-1$ must be adjacent to every vertex not specifically excluded by the adjacencies of n. In particular, $m-1$ is adjacent to both 1 and 2. If n is not adjacent to $m-2$, then $m-1$ adjacent to $m-3$ is ruled out by the minimality of r, effectively introducing another exclusion, forcing $\deg(m-1)<(n-1)/2$. From this we conclude that $m-2\in N(n)$ and $(n, m-2, m-3, \ldots, 3, 2, m-1, 1, n)$ is an m-cycle in G.

Case 2.2: When $r=2$, we consider more closely the distribution of neighbours of n in R. As in the above discussion, we see that each vertex $j\in\{m, m+1, \ldots, n-1\}$ which is adjacent to n rules out the vertex $j+1$ as a neighbour of $m-1$. Indeed, if such a vertex j is adjacent to n, then $j-1$ cannot be adjacent to $m-1$ either. Thus each neighbour of n in $\{m, m+1, \ldots, n-1\}$ excludes at least one possible neighbour of $m-1$ in $\{m, m+1, \ldots, n\}$. Furthermore, if n is not adjacent to precisely every second vertex in $\{m, m+1, \ldots, n-1\}$ (i.e. if there exists $j\in\{m, m+1, \ldots, n-1\}$ such that either n is adjacent to both j and $j+1$ or such that n is adjacent to neither of j and $j+1$), then the number of vertices in $\{m, m+1, \ldots, n\}$ which cannot be adjacent to $m-1$ is greater than the number of neighbours of n in $\{m, m+1, \ldots, n-1\}$ and, consequently, $m-1$ must be adjacent to every vertex not explicitly ruled out

[1]By the observation in each block of $p\geqslant 2$ consecutive neighbours of n excludes $p+1$ vertices from being neighbours of $m-1$ without possible double counting. Thus we have at most one block of two or more consecutive neighbours of n in $\{m, m+1, \ldots, n-1\}$. Similarly, we have at most one block of two or more consecutive nonneighbours of n in $\{m, m+1, \ldots, n-1\}$. So, in our current situation we must have precisely one block of two consecutive neighbours of n or precisely one block of two consecutive nonneighbours of n but not both. In either case there are vertices j, k in $\{m, m+1, \ldots, n-1\}$, with $|j-k|=3$, such that n is adjacent to j and $m-1$ is adjacent to k. This gives us an m-cycle $(n, 5, 6, \ldots, m-1, k, \ldots, j, n)$.

by neighbours of n as mentioned above. In particular, $m-1$ must be adjacent to the vertices 1 and $n-1$, and n must be adjacent to $m-2$ (otherwise the minimality of r excludes a further neighbour of $m-1$). Thus, neither n nor $m-1$ can have consecutive neighbours in $\{1,2,\dots,m-2\}$ (otherwise we have an m-cycle). Moreover, $m-1$ is not adjacent to either of $j+1$ and $j-1$ for n adjacent to $j\in\{1,2,\dots,m-1\}$ (otherwise we get the m-cycle $(n,j,j-1,j-2,\dots,1,m-1,j+1,j+2,\dots,m-2,n)$ or similar in the case where $m-1$ is adjacent to $j-1$). Again, if n is not adjacent to precisely every second vertex in $\{1,2,\dots,m-1\}$, then the number of vertices excluded as neighbours of $m-1$ is greater than $(n+1)/2$, which is a contradiction. Thus both n and $m-1$ are adjacent to $\{1,3,5,\dots,m-2\}$ and m is odd.

Thus we may suppose that n is adjacent to $\{m,m+2,m+4,\dots,n-1\}$ and no others in $\{m,m+1,\dots,n-1\}$ and that m is even. Now if $m-1$ is not adjacent to 1, then $N(m-1)=\{m,m+2,m+4,\dots,n-1\}\cup\{j\colon 1\leqslant j\leqslant m-2$ and n is not adjacent to $j+1\}$. Moreover, n must be adjacent to $m-2$, so that $m-3$ cannot be an extra exclusion from $N(m-1)$ by the minimality of r. Hence, $m-1$ has no consecutive neighbours in $\{1,2,\dots,m-2\}$. Consequently, $\deg(m-1)<(n-1)/2$, a contradiction. Thus we must conclude that $m-1$ is adjacent to 1. As before, we see that this means that n has no consecutive neighbours in $\{1,2,\dots,m-1\}$. Also, we know that n is not adjacent to $m-3$ because that would result in the m-cycle $(n,m-3,m-4,\dots,3,2,1,m-1,m,n)$. So n must be adjacent to $m-2$ or else $\deg(n)<(n-1)/2$. This means that the neighbours and nonneighbours of n in $\{1,2,3,\dots,m-2\}$ alternate except for one pair of consecutive nonneighbours, $j,j+1$ with $4\leqslant j\leqslant m-4$. Now we apply the arguments used above on the neighbours of m to see that m and n have no common neighbours in $\{m-1,1,\dots,n-1,n\}$ and that if n is adjacent to j in $\{1,2,\dots,m-2\}$, then m cannot be adjacent to either $j+2$ or $j-2$. From this we determine that $\deg(m)<(n-1)/2$. This contradiction concludes our consideration of case 2.2.

Case 2.3: When $r=1$ we see, as before, that every neighbour j of n, $j\in L$, excludes $j-1$ from being a neighbour of $m-1$. Furthermore, if n is adjacent to $j'\in\{m,m+1,\dots,n-1\}$, then $m-1$ cannot be adjacent to j'. Thus the neighbourhood of n completely determines the neighbourhood of $m-1$.

Similar arguments show that the neighbourhood of 1 also completely determines the neighbourhood of $m-1$ (i.e. the adjacencies of 1 rule out adjacencies of $m-1$ in exactly the same way as do those of n). This means that $N(n)\setminus\{1\}=N(1)\setminus\{n\}$. In particular, $n-1\in N(1)$.

We may now consider the neighbourhoods of $n-1$ and n to determine that $N(n-1)\setminus\{n\}=N(n)\setminus\{n-1\}$ and that $n-2\in N(n)$. Continuing in this way we find that $(i,i+2)\in E(G)$ for all $i=1,2,\dots,n-2$ and $(n-1,1),(n,2)$ are also edges in $E(G)$. But then for m odd, $(1,3,5,\dots,m,m-1,m-3,\dots,2,1)$ is an m-cycle in G, and for m even, $(1,3,5,\dots,m-1,m,m-2,m-4,\dots,2,1)$ is an m-cycle in G. Thus G is pancyclic and this completes the proof of the theorem. □

Theorem 2.3. *Let $G \in O(n-1)$. Then G is pancyclic unless it is isomorphic to one of the following graphs:*

(i) G_0 *(the n vertex graph consisting of two complete graphs joined at a point),*
(ii) *a subgraph of* $K^c_{(n+1)/2} \vee K_{(n-1)/2}$,
(iii) $K_{n/2,n/2}$
(iv) C_5.

Proof. Suppose that $G \in O(n-1)$ and that G is not (i) or (ii). Then G is 2-connected and, by Theorem 1.2, G is also hamiltonian.

Let $C = (1, 2, 3, \ldots, n, 1)$ be a hamiltonian cycle in G. We consider the following three cases:

(1) There are consecutive vertices $i, i+1$ on C such that $\deg(i) + \deg(i+1) \geqslant n+1$;

(2) case 1 does not hold but there are consecutive vertices $j, j+1$ on C such that $\deg(j) + \deg(j+1) = n$;

(3) every pair of consecutive vertices $k, k+1$ on C has $\deg(k) + \deg(k+1) \leqslant n-1$.

Case 1: By Lemma 1.4, we conclude that G is pancyclic.

Case 2: Let us assume that $\deg(1) + \deg(n) = n$. By Theorem 1.3, either

(a) G is pancyclic, in which case we are finished,

(b) G is bipartite, in which case G is either (ii) or (iii) from the statement of the theorem, or

(c) G is missing only an $(n-1)$-cycle, $\deg(n-2), \deg(n-1), \deg(2), \deg(3) < n/2$, and the vertices $n-2, n-1, n, 1, 2, 3$ are independent except for the edges of C, and $(n, n-3), (n, n-4), (1, 4), (1, 5)$.

Case 2.1: $n \equiv 0 \pmod 2$. Consider the degree of the vertex $n-2$, say. Now, Theorem 1.3(iii) states that $\deg(n-2) < n/2$. So, since n is even, $\deg(n-2) \leqslant (n-2)/2$. But then $\deg(2) \geqslant n/2$ and $\deg(3) \geqslant n/2$, contradicting Theorem 1.3(iii). Thus case 2.1 cannot occur.

Case 2.2: $n \equiv 1 \pmod 2$. We claim that each of the vertices $1, 2, 3, n-2, n-1, n$ has degree at least $(n-1)/2$. To see this, suppose that, say, $\deg(1) \leqslant (n-3)/2$. Then $\deg(n-2) \geqslant (n+1)/2$ and $\deg(n-1) \geqslant (n+1)/2$, giving $\deg(n-2) + \deg(n-1) \geqslant n+1$ and again G is pancyclic.

Similar arguments may be applied to the vertices $2, 3, n-2, n-1$ and also to vertex n if G has the first structure in Theorem 1.3(iii). If G has the second structure in Theorem 1.3(iii), then $\deg(n) > \deg(1)$ and we already know that $\deg(1) \geqslant (n-1)/2$. This establishes the claim.

Since $\deg(1) + \deg(n) = n$, we must have either $\deg(1) = (n+1)/2$ and $\deg(n) = (n-1)/2$ or $\deg(1) = (n-1)/2$ and $\deg(n) = (n+1)/2$. In the former case, $\deg(2) = (n-1)/2$ and we can apply Theorem 1.3 to vertices 1 and 2 to force the edge $(1, n-2)$ which is already known not to exist. In the latter case similar arguments may be applied unless the graph has the second structure in Theorem 1.3(iii). Here we note that $\deg(n-1) = (n-1)/2$ and apply Theorem 1.3 to the vertices $n-1$ and n. This will force an edge which is known not to exist unless the graph has the second structure in Theorem 1.3(iii) for the vertices $n-1$ and n. Given this structure, we consider the neighbours of 1. Note that 1 has degree $(n-1)/2$ and cannot be adjacent to $1, 3, n-1, n-2, n-3$. Thus 1 has $(n-5)/2$ neighbours in $\{4, 5, \ldots, n-5, n-4\}$. So 1 must have two consecutive neighbours j and $j+1$, $4 \leqslant j \leqslant n-5$, then $(1, j, j-1, j-2, \ldots, 4, 3, n, n-1, n-2, \ldots, j+1, 1)$ is an $(n-1)$-cycle in G, a contradiction.

Case 3: $\deg(i) + \deg(i+1) \leqslant n-1$ for all $1 \leqslant i \leqslant n$ (where $n+1 \equiv 1$). If every vertex has degree $(n-1)/2$, then G is $(n-1)/2$-regular and, by Theorem 2.2, we are done. Thus we may assume that there is a vertex $u \in V(G)$ with $\deg(u) < (n-1)/2$. Say $\deg(u) = (n-p-1)/2$ where $p > 0$.

Since $G \in O(n-1)$, every vertex $v \in V(G)$ with $\deg(v) < (n+p-1)/2$ is adjacent to u. There are at most $(n-p-1)/2$ such vertices. Thus there are at least $n-(n-p-1)/2-1 = (n+p-1)/2$ vertices with degree at least $(n+p-1)/2 \geqslant n/2$. This, together with parity considerations, implies that there are two consecutive vertices on C with degree sum at least n unless $p=1$ and n is even. But in this case we see that G contains a set of $n/2$ mutually adjacent vertices, each of which has degree $(n-2)/2$, i.e. G is disconnected. This contradiction completes the proof of the theorem. $\square$

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